American Journal of Theoretical and Applied Statistics
2021; 10(6): 257-261
http://www.sciencepublishinggroup.com/j/ajtas
doi: 10.11648/j.ajtas.20211006.15
ISSN: 2326-8999 (Print); ISSN: 2326-9006 (Online)

On Approximation of Copulas, Solution of Elliptic Problem, Using the Finite Difference Method: Generalization

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To cite this article:
Remi Guillaume Bagré, Frédéric Béré, Kpèbbèwèrè Cédric Somé, Paliguiwindé Dieudonné Ibrango. On Approximation of Copulas, Solution of Elliptic Problem, Using the Finite Difference Method: Generalization. *American Journal of Theoretical and Applied Statistics*. Vol. 10, No. 6, 2021, pp. 257-261. doi: 10.11648/j.ajtas.20211006.15

Received: October 26, 2021; **Accepted**: November 26, 2021; **Published**: December 9, 2021

Abstract: In this paper, we generalize a copula construction method discussed in one of our papers. For this purpose we consider the general form of a linear elliptic PDE. Indeed, a physical interpretation of elliptic equations comes from the notion of conservative flow given by a gradient. This notion provides a mathematical model for equilibrium conservation laws in linear behaviour. This can be applied to many areas of science. Thus, the aim of this paper is to construct a new class of bivariate copulas by solving an elliptic partial differential equation with a Dirichlet condition at the boundary. Copulas belonging to this class allow us to study the stochastic behaviour of the notion of conservative flows. In other words, these copulas will allow us to have an idea on the dependence of those physical phenomena which are governed by elliptic PDEs. For this purpose, we use a discretization method which is the finite difference method which is a common technique for finding approximate solutions of partial differential equations that consists in solving a system of relations (numerical scheme) connecting the values of the unknown functions at some points sufficiently close to each other. For the finite difference method, a mesh is made. This is a set of isolated points called nodes located in the domain of definition of the functions subject to the partial differential equations, a grid on which only the nodes of which the unknowns corresponding to the approximate values of these functions are defined. The mesh also includes nodes located on the boundary of the domain (or at least "close" to this boundary) in order to be able to impose the boundary conditions and/or the initial condition with sufficient accuracy. The primary quality of a mesh is to cover the domain in which it develops as well as possible, to limit the distance between each node and its nearest neighbour. However, the mesh must also allow the discrete formulation of the differentiation operators to be expressed: for this reason, the nodes of the mesh are most often located on a grid whose main directions are the axes of the variables. In the main results of this paper (see section 3), we give a discretization of the solution of the problem followed by a simulation with the MATLAB software of this approximated solution and presenting the discretization errors.

Keywords: Copulas, Linear Elliptic PDE, Boundary Value Problem, Finite Difference Method, Discretization

1. Introduction

As announced in the introduction, this article draws on several works (see [6, 8, 10, 11, 14, 15]). In these different works we have some hints for the construction of copulas that we try to generalize in this paper. Copulas thus appear to be a natural tool for constructing multivariate distributions via Sklar's theorem when the marginal distributions are sfficiently regular. Sklar's theorem thus provides a canonical representation of a multivariate distribution, via the data of marginal distributions and dependency structure. The definition and properties of copulas are illustrated in [9]. In this paper, we will limit ourselves to two variables (dimension 2) for the sake of clarity and conciseness. In [10], We had already constructed a family of copulas using finite differences. This family being solution of a particular elliptic PDE (see [2]). But in this article, the copula family is a solution of an elliptic PDE in the general case.

The aim of this paper is to construct a family of copulas C solution of an elliptic boundary value problem based on given finite diference scheme. Thus, this new class of copula solution of the boundary value problem is given by the following problem:

$$(P):\begin{cases} -div\left(grad\big(C(u,v)\big)\right) + \alpha C_u(u,v) + \\ \beta C_u(u,v) + \gamma C_u(u,v) \\ = f(u,v)\ (u,v)\epsilon[0;1]^2 \\ C(u,0) = 0 = C(0,v) \\ C(u,1) = u \text{ and } C(1,v) = v \end{cases} \tag{1}$$

where α, β, γ are real number and C is a copula. f is a function defined on $[0;1]^2$. We notice that $div\big(grad(.)\big) = \nabla^2(.)$. C_u is gradient of C in u direction and C_v is gradient of C in v direction (see [7]).

The boundary value problem described above can be thought of as a non-homogenous Dirichlet problem. The solution to this problem is not analytically known in general. An approximation is then made to reduce the problem to a finite number of unknowns (discretization process). We therefore introduce a mesh of steps h in the direction u and v. The nodes of the mesh are the points $P_{ij} = (u_i, v_j)$ where the solution is approached. The following can be noted :

$u_i = ih, 0 \le i \le N+1$, the vertices of the mesh in the u-direction.

$v_j = jh, 0 \le j \le N+1$, the vertices of the mesh in the v-direction.

We are looking for an approximation of the equation at the nodes of the mesh: $(P_{ij}, 1 \le i \le N, 1 \le j \le N)$. The principle of the finite diference method consists in approximating the derivatives of a function by linear combinations of the values of this function at the points of the mesh.

2. Preliminaries

In the rest of the document, we will note $I = [0,1]$ and $I^2 = I \times I = [0,1] \times [0,1]$.

Definition 1 (see [9, 12])

A copulas is a function $C: I^2 \to I$ with the following properties:

1. For every $u, v \epsilon I$,
   (1) $C(u,0) = 0 = C(0,v)$
   (2) $C(u,1) = u$ and $C(1,v) = v$
2. C is 2-increasing, i.e for every u_1, u_2, v_1, v_2 *in* I such that $u_1 \le u_2$ and $v_1 \le v_2$, we have (3) $V_H(B) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0$ where B is the rectangle $[u1; u2] \times [v1; v2]$ and the expression (3) defines the C-volume of B.

According to sklar theorem (see [9], Theorem 2.3.3) with given $F; G$ and H defned as above, there exists a copula C such that for all $x;\ y \,\epsilon\, \overline{\mathbb{R}}$,

$$H(x;\ y) = C(F(x); G(y)) \tag{2}$$

and conversely, for any copula C, the function H defined with (2) is a joint distribution function with margins $F(x)$ and $G(y)$. The aim of this paper is to construct a family of copulas C using the concept of finite diference schemes.

The finite diference method is one of the oldest methods of numerical simulation which is still used for some applications. This method appears to be the simplest to implement because it proceeds in two steps: on the one hand the discretization by finite diferences of the diferentiation operators, and on the other hand the convergence of the numerical scheme thus obtained when the distance between the points decreases.

Definition 2 (see [4, 5, 13])

We called error of consistency of the numerical scheme $A_h u_h = b_h$ the vector $\mathcal{E}_h(u)\epsilon\mathbb{R}^N$ defined by:

$$\mathcal{E}_h(u) = A_h\big(\Pi_h(u)\big) - b_h \text{ where } \Pi_h(u) = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix} \tag{3}$$

$\Pi_h(u)$ represents the projection of the exact solution onto the mesh. The scheme is said to be consistent for norm $||.||$ of $\mathbb{R}^N$ if $\lim_{h\to 0} \mathcal{E}_h(u) = 0$. If moreover there is a constant α independent of h such that:

$$\big||\varepsilon_h(u)|\big| \le \alpha h^p \tag{4}$$

$\forall\, p > 0$, the schema is said to be of order p for the norm $||.||$.

Usually we use the norms (see [1, 3, 15])

$$||.|| = ||.||_1, ||.||_2 \text{ or } ||.|| = ||.||_\infty.$$

Definition 3 (see [4, 5, 13])

We will say that a scheme is stable for the norm $||.||_\infty$, there exist a constant $\alpha > 0$ independant of h such that:

$$||u_h||_\infty = \text{Sup}_i\, |u_i| \le \alpha \tag{5}$$

3. Main Results

Proposition 1 Let C, a copula, be the exact solution of the problem (1). Let be $n \in \mathbb{N}$, we pose $h = \frac{1}{N+1}$ and $C_{i,j}$ is the desired approximate copulas of $C(ih, jh), (i,j) \in \{1, \dots, N\}^2$. We pose $f_{i,j} = f(ih, jh), \forall, (i,j) \in \{1, \dots, N\}^2$.

$(\mathcal{P})$ can be written as follows:

$$(\mathcal{P}_h):\begin{cases}\alpha_0 C_{i,j}-\alpha_1 C_{i-1,j}-\alpha_2 C_{i+1,j}-\alpha_3 C_{i,j-1}-\alpha_4 C_{i,j+1}=f_{i,j}\\ C_{i,j}-C_{i,j-1}-C_{i-1,j}+C_{i-1,j-1}\geq 0\\ C_{0,j}=0=C_{i,0}\\ C_{i,N+1}=\dfrac{i}{N+1}\ and\ C_{N+1,j}=\dfrac{j}{N+1}\end{cases} \tag{6}$$

Where

$$\alpha_0=\frac{4}{h^2}+\gamma, \alpha_1=\frac{1}{h^2}+\frac{\alpha}{2h},\ \alpha_2=\frac{1}{h^2}-\frac{\alpha}{2h}, \alpha_3=\frac{1}{h^2}+\frac{\beta}{2h} \text{ and } \alpha_4=\frac{1}{h^2}-\frac{\beta}{2h} \tag{7}$$

Proof. Each derivative is discretized according to its own direction, so by applying Taylor's formula in the u and v directions, we have:

$$C(u+h,.)=C(u,.)+hC_u(u,.)+\frac{h^2}{2}C_{uu}(u,.)+\vartheta(h^2) \tag{8}$$

and

$$C(u-h,.)=C(u,.)+hC_u(u,.)+\frac{h^2}{2}C_{uu}(u,.)+\vartheta(h^2) \tag{9}$$

By summing up (8) and (9) we get:

$$C_{uu}(u,.)\simeq\frac{C(u+h,.)-2C(u,.)+C(u-h,.)}{h^2} \tag{10}$$

The relation (10) can be approached as follows:

$$C_{uu}\simeq\frac{C_{i+1,j}-2C_{i,j}+C_{i-1,j}}{h^2} \tag{11}$$

By a similar reasoning we obtain that:

$$C_{vv}\simeq\frac{C_{i,j+1}-2C_{i,j}+C_{i,j-1}}{h^2} \tag{12}$$

Let us now write second order approximation approximation of $Cu(u;\ v)$, By performing the subtraction between the relation (8) and (9) we obtain:

$$C_u(u,.)\simeq\frac{C(u+h,.)-C(u-h,.)}{2h}\simeq\frac{C_{i+1,j}-C_{i-1,j}}{2h} \tag{13}$$

and

$$C_v(.,v)\simeq\frac{C(.,v+h)-C(.,v-h)}{2h}\simeq\frac{C_{i+1,j}-C_{i-1,j}}{2h} \tag{14}$$

By summing up the relations (11), (12), (13), (14) we obtain the desired results.

4. Simulations

We make in this part a simulation of the approached solution using the MATLAB environment.

The figures below will concern:

1 The numerical solution of the approximate copula C by finite diference.

2 The Error of the discretization of the approximate copula.

1) Let's suppose that $f(u;\ v)\ =\ u\ +\ v\ +\ uv$, we get the following figures:

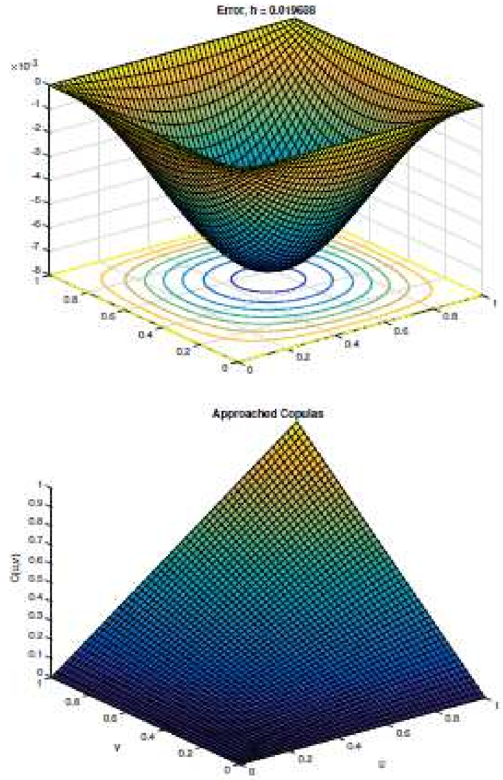


Figure 1. *Approached solution when* $N\ =\ 50$ *and* $\alpha=\beta=\gamma=1$.

The Figure 1 give us the approached solution of finite diference method (at Right) and the error of the estimation of our copulas (at Left).

2) Let's suppose now f is the independant copula i.e $f(u;\ v)\ =\ uv$ then get the following Figures:

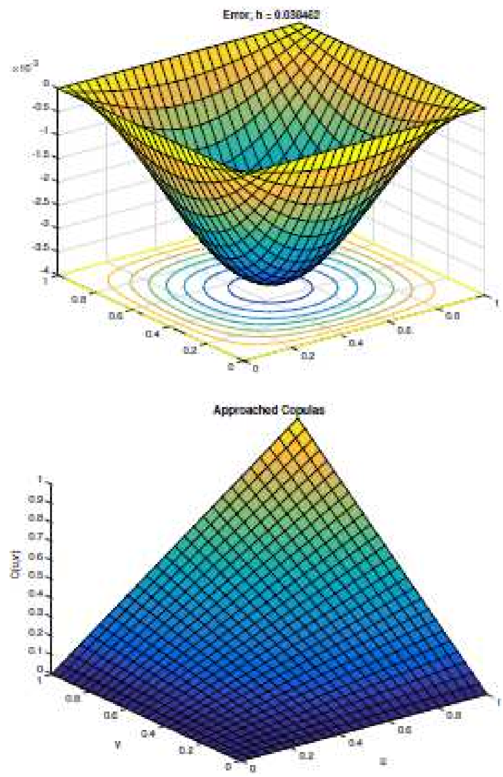
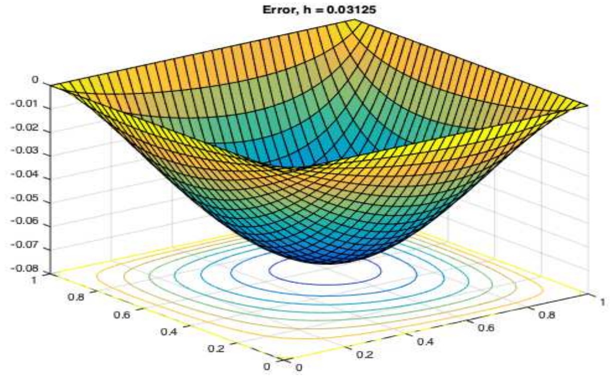


Figure 2. *Approached solution when* $N\ =\ 25$ *and* $\alpha=\beta=0$ *and* $\gamma=1$.

The Figure 2 give us the approached solution of finite diference method when $N = 25$ and the error of the estimation of our copulas.

3) Let's suppose that $\alpha = 1, \beta = 1, \gamma = 0$ and $f(u,v) = u + v$, so the approched copulas and the the error of the dicretization is given by the following figures:



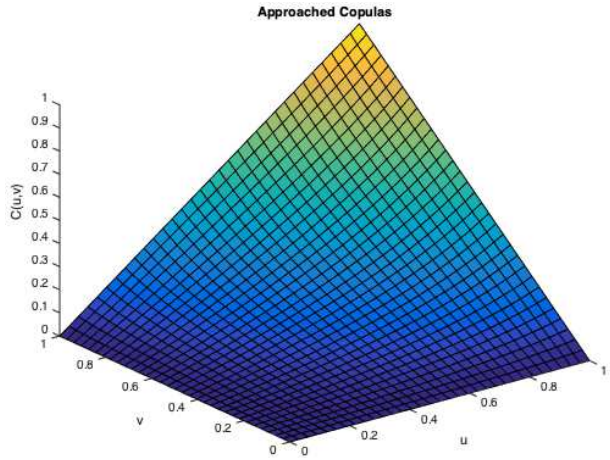
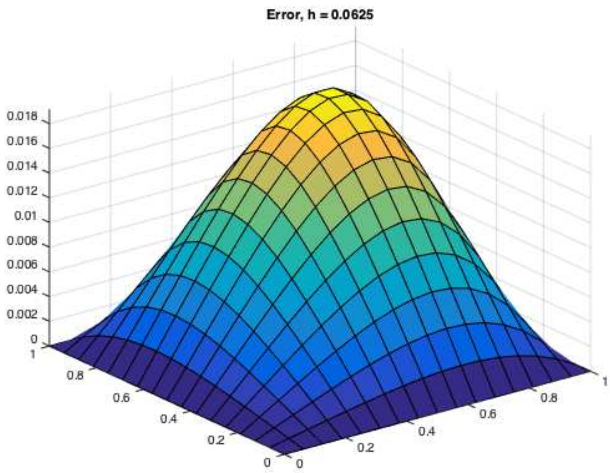


Figure 3. Approached solution when $N = 32$ and $\alpha = \beta = 1$ and $\gamma = 0$.

4) Now let's suppose that $\alpha = \beta = \gamma = 0$ and $f(u,v) = uv$, the independent copulas, so the approched copulas and the error of the dicretization is given by the following figures:



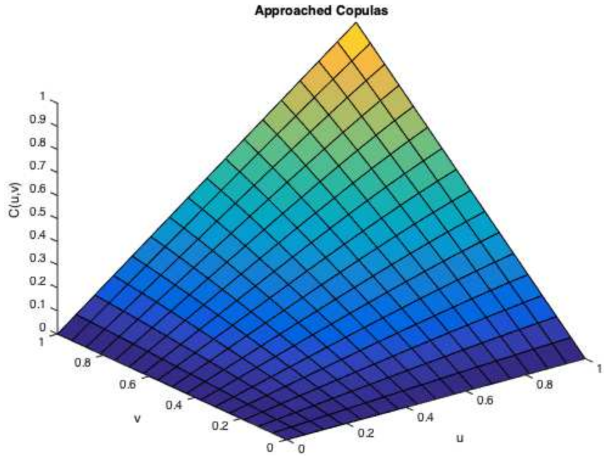
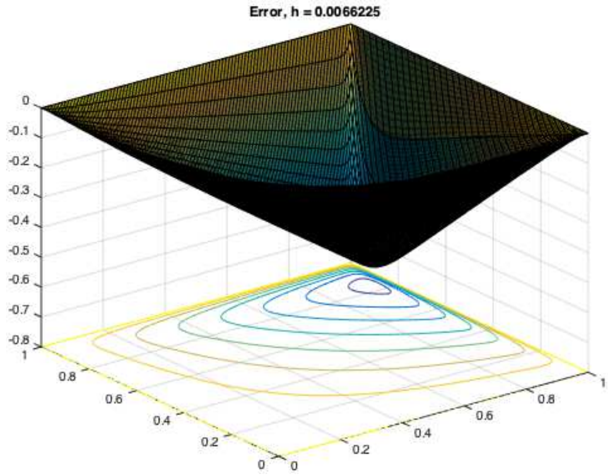


Figure 4. Approached solution when $N = 15$ and $\alpha = \beta = \gamma = 0$.

5) Let's suppose that $\alpha = 10, \beta = 75, \gamma = 50$ and $f(u,v) = \min(u,v)$, so the approched copulas and the the error of the dicretization is given by the following figures:



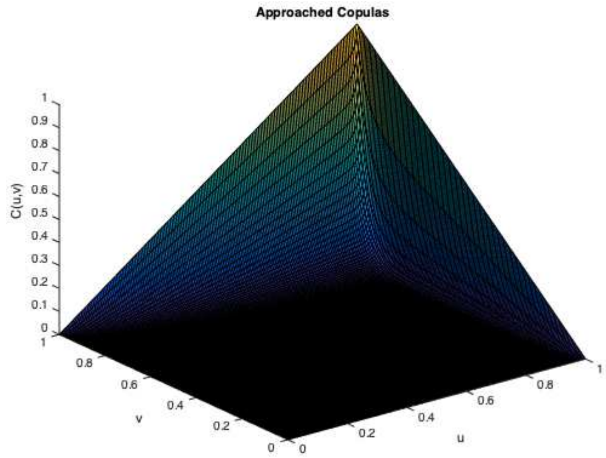


Figure 5. Approached solution when $N = 150$ and $\alpha = 10, \beta = 75, \gamma = 50$.

6) Let's suppose that f is a Gumbel Copula i.e $f(u,v) = e^{\left(-\left[(-\ln u)^{\theta}+(-\ln v)^{\theta}\right]^{1/\theta}\right)}$ (u,v) in I^2 and $\theta \geq 1$. We get the following figures:

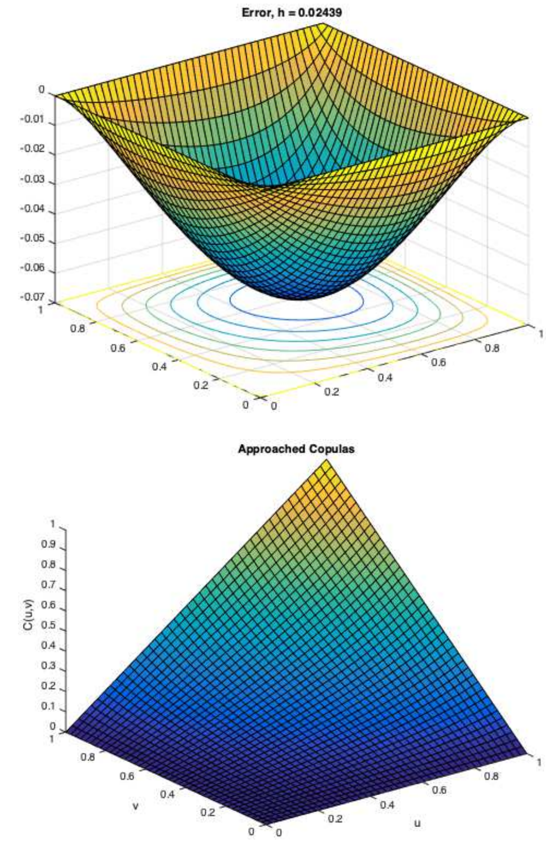


Figure 6. *Approached solution when* $N = 40$ *and* $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\theta = 3$.

When the infinite norm is not bounded then the relation (6) is not convergent i.e. (6) is neither stable nor consistent. This can be seen in the following figure:

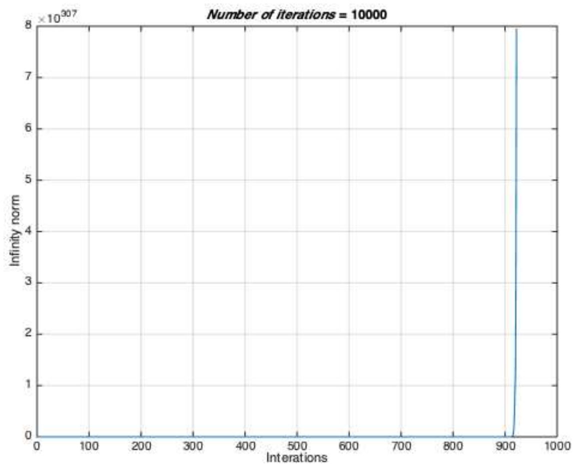


Figure 7. *Gauss-Seidel iterations when* $N = 10$ *and* $\alpha = 10$, $\beta = 75$, $\gamma = 50$.

In Figure 7, it is easy to see that $||C||_{\infty} \to +\infty$. so C is not bounded.

5. Conclusion

In this paper, we are inspired by paper [10], but generalize it by constructing a family of copulas as a solution of a linear elliptic PDE., by the finite difference method. This method gives an approached solution which converges to the exact solution of the boundary value problem. We use Matlab environment to make numerical simulations.

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